

Math is more than Computation: Teach Algorithms for Deeper Understanding

Place value is the underlying concept of our number system, the reason many algorithms work so simply: the marvelous invention of having each column worth ten times the one to its right. It took humans thousands of years to come up with and then accept this system, so it is no wonder that it takes children a long time to fully appreciate it. It is a topic that we need to revisit every year, so that students come to understand it in different and more profound ways. Play place value games such as Double Digit from *Family Math* by Jean Kerr Stenmark, Virginia Thompson, Ruth Cossey, and Marilyn Hill; (CA:Equals Series, Paperback, 1986); make the largest sum of 2-digit numbers from 4 randomly selected 0–9 cards; make the largest number possible from 4 randomly selected 0–9 cards.

Choose algorithms by their relation to the concepts you want to teach.

Subtraction

Difference, how far apart 2 numbers are, is easily seen for small differences, on a number line. 2 to 5, 3 to 6, 7 to 10 are equal distances apart. We can use this idea of “constant difference” to perform subtraction by sliding up:

$$\begin{array}{r} 346 \\ -267 \\ \hline \end{array} \quad +3 \quad \rightarrow \quad \begin{array}{r} 349 \\ -270 \\ \hline \end{array} \quad +30 \quad \rightarrow \quad \begin{array}{r} 379 \\ -300 \\ \hline \end{array}$$

or down:

$$\begin{array}{r} 400 \\ -137 \\ \hline \end{array} \quad -1 \quad \rightarrow \quad \begin{array}{r} 399 \\ -136 \\ \hline \end{array}$$

You can *assess students’ understanding of constant difference* by asking:

Jordan and Taylor both had baseball card collections. One day they counted them and Jordan had 53 more than Taylor. Then Jordan’s brother said he didn’t want his, and they could have them. They each got 83 more cards. Now does one of them have more than the other? If so, how many more does she have?

Removal, taking away, and comparison provide an opportunity to focus on *place value* with a “regrouping” algorithm. Students must be able to see a number such as 400 as both “4 hundreds” and as “3 hundreds, 9 tens, and 10 ones”. Errors in subtraction are usually caused by not making sure that the minuend remains the same quantity when it is rewritten for an algorithm. Precede work with a place-value-based algorithm by having the students find all the ways to make a number in hundreds, tens and ones, also encouraging pattern-finding and making an orderly list.

Subtraction as the addition of the additive inverse of a positive number (as in $8 + -5$) prepares students for *integer arithmetic and algebraic notation*:

$$\begin{array}{r} 325 \\ -187 \\ \hline 2-6-2 \end{array} \quad \begin{array}{r} 3x^2 + 2x + 5 \\ - (x^2 + 8x + 7) \\ \hline 2x^2 - 6x - 2 \end{array}$$

which means 2 hundreds – 6 tens – 2 ones

$$= 200 - 60 - 2 = 138$$

and relates to polynomial subtraction as shown above on the right.

Addition

Base ten materials can help students visualize *place value*, but should not be used repeatedly to perform arithmetic. Students might use them to check their work. If the answers are different, can they explain what happened? Do they know which answer is correct? Error analysis is another math skill students need to develop for their life, not just class work.

The regrouping algorithm does not prepare students for *polynomial addition*: $x + x^2 = x + x^2$. Base ten materials can demonstrate, for example $253 + 478$: If each of these numbers is built with base ten materials, how many hundreds, tens and ones will I need?

Multiplication combines many arithmetic properties

When we multiply

$$\begin{array}{r} 341 \\ \times 27 \\ \hline 2387 \\ 6820 \\ \hline \end{array}$$

We may be using as many as 5 properties:

1) The *distributive property* of multiplication over addition: the ability to split numbers into addends, as when multiplication by 27 is treated as multiplication by 7 and by 20.

2) The *fundamental theorem of arithmetic*: We can treat the multiplication by 20 as multiplication by 10 (writing the 0 in the ones place) and then by 2 because $20 = 10 \times 2$.

Expansion of this idea leads to appreciation of the role of prime numbers.

$$2^3 \times 3^2 \times 7 = 504 \quad \text{Is 17 a factor of 504?}$$

I pose this question and see if listeners reach for a calculator or pencil and paper to find out. If they do, I ask them to find all the factor pairs for 504. Once they have found them, they are asked to find the prime factors of each of those factors, stopping after a few to make observations about what they have noticed: only 2, 3, & 7 ever appear and each factor pair contains exactly three 2s, two 3s and one 7. This leads to the *Fundamental Theorem of Arithmetic*: Every natural, or counting, number greater than one is either a prime number or a product of powers of prime numbers. These are **unique to that number**, except for the order in which they are written. In other words, every factor of a number is one of the primes in its prime factorization or a combination of those primes. No other prime numbers can sneak their way in.

3) *Place value*: The effect on the powers of ten as we multiply: 20×40 gives us 8 hundreds. . . A whole class project to create a place value chart can help students appreciate the vast increase as the powers of 10 increase. It should contain the numbers written in words, standard notation, and exponential form with columns showing every way to create each place as a product of powers of 10.

4) *associative property* of multiplication: $341 \times 20 = 341 \times (10 \times 2) = (341 \times 10) \times 2$

5) *commutative property* of multiplication: 20 means 2×10 but we multiply by 10 first.

Polynomial multiplication when students know that the powers of 10 multiply as: $10^3 \times 10^2 = 10^5$, they are better prepared to understand that $X^3 \cdot X^2 = X^5$.

Division

The so-called “partial quotient” method emphasizes the *relationship between multiplication and division* and the meaning of *remainder*. An activity that focuses on the meaning of remainder:

The Candy Factory

I have a candy factory where we make very special, fancy, decorated chocolates. We only make 100 of these each day. We sell these chocolates different ways. Sometimes we have little boxes with just one candy in each box. On other days we put two candies in each box. We also have boxes for three candies, four candies, and so on, up to a huge box with 100 candies in it! We only fill one kind of box each day. We need you to figure out how many boxes we need to have ready each day.

Each day we start with 100 candies. On the first day we put one candy in a box. On the second day we put two candies in a box. On the third day we put three candies in a box. We continue in this way until, on the hundredth day we put 100 candies in a box. (Then we start over with one candy in a box.)

Here is a table for entering the information about the candies. Fill in the table for 100 days.

Packing 100 candies into different-size boxes		
number of candies in each box	number of boxes we need	number left over
1		
2		
3		
•		
•		
•		
100		

(For extra thinking:)

If you got to keep all the leftovers, how many candies would you get in 100 days?

Is it possible, on some days, to have more candies left over than the number of boxes you have filled? Explain your answer.

Is it possible to have more candies left over than you put into each box? Explain your answer.

There are many patterns that can be found in the table. Describe the ones you have found. Then look at the table again, find some more patterns and write about them too.

To *assess understanding of remainder* ask:

I divided a number by 7 and got a remainder of 5. The number is between 50 & 100.

What numbers could have been the dividend – the number I started with?

The traditional long division algorithm also relies on partial quotients and remainder, but requires a deeper understanding of *place value*. $400 \div 7$ requires seeing 400 as 40 tens and then 5 tens as 50 ones. Taught with that in mind and sufficient place value activities beforehand, that algorithm can provide a good way to learn about and increase your students’ knowledge of place value.